MODIFIED NOVIKOV–VESELOV EQUATION AND DIFFERENTIAL GEOMETRY OF SURFACES

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1. INTRODUCTION

In the present paper we consider global soliton deformations of surfaces immersed in the three-dimensional Euclidean space.

Local deformation of surfaces represented via the generalized Weierstrass formulas (3.3 - 3.4) were introduced by Konopelchenko ([Kon]) by using of the modified Novikov-Veselov equation (2.11) which in turn was introduced by Bogdanov [Bg1]). This equation is a modification of the Novikov–Veselov equation in the same sense as the modified Korteweg–de Vries equation is a modification of the Korteweg–de Vries equation. Saying about the geometric meaning of the Novikov–Veselov equation we notice that it has important applications to the theory of algebraic curves ([T,Sh]).

Here we will discuss applications of the modified Novikov–Veselov equation to differential geometry of surfaces. Investigation of global properties of mNV deformations of surface in the case of tori of revolution was started in [KT] where it was particularly shown that tori of revolution are preserved by these deformations. But at that time the relation of these deformations to conformal geometry was not understood.

In the present paper we consider global deformations of surfaces of general type and it's relation to the theory of the Willmore functional which is defined as integral of squared mean curvature (5.1).

Locally any regular surface is represented via the generalized Weierstrass formulas and moreover any analytic surface is globally represented in this manner (see Propositions 1 and 2). Thus at least for analytic surfaces the mNV–deformation is correctly defined.

We show that

mNV-deformation transforms tori into tori and preserves their conformal structure and value of the Willmore functional (Theorems 1 and 2).

We also consider the following conjecture

Conjecture. *Non-stationary, with respect to mNV-deformation, torus can not be a local minimum of the Willmore functional*

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2

and discuss it's relation to the famous Willmore conjecture (see chapter 5). We thank M.V. Babich, B.G. Konopelchenko, and S.V. Manakov for helpful conversations.

2. Modified Novikov-Veselov equation

2.1. Novikov-Veselov equation.

The Novikov-Veselov equation (NV)

$$U_t = \partial^3 U + \bar{\partial}^3 U + \partial(VU) + \bar{\partial}(\bar{V}U), \tag{2.1}$$

$$\bar{\partial}V = 3\partial U$$

was introduced by Novikov and Veselov in [VN1] within frames of development of the theory of two-dimensional potential Schrödinger operators which are finite-zone on one level of energy ([DKN,VN2]).

Here functions U and V are defined on the complex plane and partial derivatives $\partial, \bar{\partial}$ are defined by usual formulas

$$\partial = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}), \quad \bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}), \quad z = x + iy \in \mathbf{C}.$$

This equation is a natural two-dimensional generalization of the famous Korteweg-de Vries equation (KDV)

$$U_t = \frac{1}{4}U_{xxx} + \frac{3}{2}UU_x, \tag{2.2}$$

to which the NV equation reduces, after suitable renormalization of variable x, in the case when the function $U(z,\bar{z})$ does not depend on variable y. Investigation of the KDV equation and discovery of it's prominent properties were starting points of intensive development of the soliton theory. Though the theory of this equation is well-known due to a lot of monographs (see, for instance [N]) we will dwell on some facts which are substantial for our exposition.

The KDV equation has a representation as a condition of commutativity

$$[L^{KDV}, \frac{\partial}{\partial t} - A] = 0, \tag{2.3}$$

of two scalar differential operators

$$L^{KDV} = \frac{\partial^2}{\partial x^2} + U, \tag{2.4}$$

$$A = \frac{\partial^3}{\partial x^3} + \frac{3}{2}U\frac{\partial}{\partial x} + \frac{3}{4}U_x.$$

In such case we call that equation is represented by L, A-pair.

The Novikov-Veselov equation as against of the KDV equation is represented by L, A, B-triple

$$\frac{\partial L}{\partial t} + [L, A] - BL = 0 \tag{2.5}$$

where
$$L^{NV} = \partial\bar{\partial} + U, \tag{2.6}$$
$$A = (\partial^3 + V\partial) + (\bar{\partial}^3 + \bar{V}\bar{\partial}),$$

and
$$B = \partial V + \bar{\partial}\bar{V}.$$

Here B is the scalar operator of multiplication by function.

The representation of nonlinear equations by L, A, B-triples was introduced by Manakov ([M]) as a two-dimensional generalization of representation (2.3). Indeed, equation (2.3) preserves spectrum of operator L and deforms it's eigenfunctions as follows :
$$\frac{\partial\psi}{\partial t} = A\psi, \quad L\psi = \lambda\psi. \tag{2.7}$$

Operator (2.6) is multidimensional and it's eigenspaces, which correspond to fixed eigenvalues, are generally infinite-dimensional . Equation (2.5) deforms not all eigenspaces but only the kernel of operator L via the equation

$$\frac{\partial\phi}{\partial t} = A\psi, \quad L\phi = 0. \tag{2.8}$$

In some sense the Novikov-Veselov equation is more natural two-dimensional generalization of the KDV equation than the famous Kadomtsev-Petviashvili equation
$$(U_t - \frac{1}{4}(U_{xxx} + 6UU_x))_x = \frac{3}{4}U_{yy}.$$

This equation also reduces to the KDV equation in the case when the function U does not depend on the space variable y and it is represented by the L, A-pair where the operator L has the form

$$\frac{\partial}{\partial y} - \frac{\partial^2}{\partial x^2} - U(x, y).$$

This operator differs from two–dimensional operator (2.6) which is a usual two-dimensional Schrödinger operator, i.e. the most natural two-dimensional generalization of the one–dimensional Schrödinger operator (2.4).

Notice that there exist two different deformations, of operator (2.6), which have form (2.5):
$$\frac{\partial L^{NV}}{\partial t^{\pm}} + [L^{NV}, A^{\pm}] - B^{\pm}L = 0.$$

These deformations are represented by L, A, B-triples for operators

$$A^+ = \partial^3 + V\partial, \bar{\partial}V = 3\partial U, B^+ = \partial V$$

and
$$A^- = \bar{\partial}^3 + \bar{V}\bar{\partial}, \partial\bar{V} = 3\bar{\partial}U, B^+ = \bar{\partial}\bar{V}.$$

But these deformations do not preserve real potentials. In it's turn equation (2.1), which, in fact, is their linear superposition, transforms real potentials U into real ones.

2.2. Modified Novikov-Veselov equation.

There exists another famous integrable $1 + 1$-dimensional integrable equation which is called modified Korteweg-de Vries equation (mKDV) :

$$U_t = U_{xxx} + 24U^2 U_x. \tag{2.9}$$

This equation is represented by L, A-pair which we will not give here. We only mention that L–operator has the following form

$$L^{mKDV} = \frac{\partial}{\partial x} - \frac{1}{2}\begin{pmatrix} -1 & 4U \\ -4U & 1 \end{pmatrix}. \tag{2.10}$$

Bogdanov introduced in [Bg1] two-dimensional generalization of the mKDV equation – the modified Novikov-Veselov (mNV) equation

$$U_t = (U_{zzz} + 3U_z V + \frac{3}{2}UV_z) + (U_{\bar{z}\bar{z}\bar{z}} + 3U_{\bar{z}}\bar{V} + \frac{3}{2}U\bar{V}_{\bar{z}}) \tag{2.11}$$

where

$$V_{\bar{z}} = (U^2)_z. \tag{2.12}$$

This equation is also, as the NV equation, a linear superposition of two deformations of form (2.5) which are represented by L, A, B-triples with common operator L defined by

$$L^{mNV} = \begin{pmatrix} \partial & -U \\ U & \bar{\partial} \end{pmatrix}, \tag{2.13}$$

and the following A- and B-operators

$$A^+ = \partial^3 + 3\begin{pmatrix} 0 & -U_z \\ 0 & V \end{pmatrix}\partial + \frac{3}{2}\begin{pmatrix} 0 & 2UV \\ 0 & V_z \end{pmatrix}, \tag{2.14}$$

$$B^+ = 3\begin{pmatrix} 0 & U_z \\ -U_z & 0 \end{pmatrix}\partial + 3\begin{pmatrix} 0 & -UV \\ -U_{zz} - UV & 0 \end{pmatrix}$$

and

$$A^- = \bar{\partial}^3 + 3\begin{pmatrix} \bar{V}\bar{\partial} & 0 \\ U_{\bar{z}} & 0 \end{pmatrix}\bar{\partial} + \frac{3}{2}\begin{pmatrix} \bar{V}_{\bar{z}} & 0 \\ -2U\bar{V} & 0 \end{pmatrix}, \tag{2.15}$$

$$B^- = 3\begin{pmatrix} 0 & U_{\bar{z}} \\ -U_{\bar{z}} & 0 \end{pmatrix}\bar{\partial} + 3\begin{pmatrix} 0 & U_{\bar{z}\bar{z}} + U\bar{V} \\ U\bar{V} & 0 \end{pmatrix}.$$

These triples represent equations

$$U_{t^+} = U_{zzz} + 3U_z V + \frac{3}{2}UV_z$$

and

$$U_{t^-} = U_{\bar{z}\bar{z}\bar{z}} + 3U_{\bar{z}}\bar{V} + \frac{3}{2}U\bar{V}_{\bar{z}}$$

where the function V is defined by formula (2.12).

Analogously to the case of the NV equation, we can derive that

1) if the function U depends only on one space variable x when modified NV equations reduce to the mKDV equation ;

2) equation (2.11) transforms real potentials into real ones as against to equations represented by $L^{mNV}, A^{\pm}, B^{\pm}$-triples ;

3) modified Novikov-Veselov equations deform the kernel of operator L via the equations

$$\frac{\partial \psi}{\partial t^{\pm}} = A^{\pm}\psi, \quad L^{mNV}\psi = 0, \tag{2.16}$$

and deformation of eigenfunctions of L^{mNV} via (2.11) is defined by

$$\frac{\partial}{\partial t}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \left(\partial^3 + \bar{\partial}^3 + 3\begin{pmatrix} 0 & -U_z \\ 0 & V \end{pmatrix}\partial + 3\begin{pmatrix} \bar{V}\bar{\partial} & 0 \\ U_{\bar{z}} & 0 \end{pmatrix}\bar{\partial}+\right.$$

$$\left.\frac{3}{2}\begin{pmatrix} 0 & 2UV \\ 0 & V_z \end{pmatrix} + \frac{3}{2}\begin{pmatrix} \bar{V}_{\bar{z}} & 0 \\ -2U\bar{V} & 0 \end{pmatrix}\right)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{2.17}$$

2.3. Hierarchies of equations.

One of outstanding properties of equations integrable by the inverse scattering method is that they are included into hierarchies of such equations which are recursively defined.

For instance, the KDV equation and it's modification (mKDV) are only first members (for $k = 1$) of hierarchies of equations of the form

$$U_{t_{2k+1}} = N_{2k+1}(U)$$

where $N_{2k+1}(U)$ are nonlinear operators. These equations are represented by L, A–pairs with operators L^{KDV} and L^{mKDV} respectively. For the KDV hierarchy operators A have the following form

$$A_k^{KDV} = \frac{\partial^{2k+1}}{\partial x^{2k+1}} + ...$$

where we denote by dots terms of lower orders. These terms are defined by condition that commutators of operators L and A would be operators of multiplication by scalars.

Thus we can say that the KDV hierarchy is attached to the operator L^{KDV}. Analogously the mKDV hierarchy is defined.

The NV equation and it's modification are also included in hierarchies for which A-operators take forms

$$A_k^{NV} = \partial^{2k+1} + ...$$

and

$$A_k^{mNV} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\partial^{2k+1} + ...$$

respectively.

We also can say that these hierarchies are attached to operators L^{NV} (i.e., the two-dimensional Schrödinger operator) and L^{mNV} (i.e., the Dirac operator), respectively.

In the soliton theory the method of defining of hierarchies by using of so-called recursion operators is well-known. For instance, the k-th equation of the KDV hierarchy takes the form

$$U_{t_{2k+1}} = R^k(U_x)$$

where the recursion operator is given by

$$R = \frac{\partial^2}{\partial x^2} + 3U + 3U_x \left(\frac{\partial}{\partial x}\right)^{-1}.$$

Regretfully, $2+1$-equations and the NV equations among them do not admit such simple representation in terms of local operator R. There exists method based on using of bilocal operators ([FS]) but it's realization is more difficult than in the case of the KDV equations.

However one can be confirmed in that by forms of higher equations. We will describe, for instance, only second equations of these hierarchies.

The NV2 equations

$$U_{t_3^+} = \Phi_{NV2}(U), \quad U_{t_3^-} = \overline{\Phi_{NV2}(U)}$$

where

$$\Phi(U)_{NV2} = \partial^5 U + V\partial^3 U + 2V_z\partial^2 U + (W+V_{zz})\partial U + W_z U = \partial^5 U + \partial(V\partial^2 U + V_z\partial U + WU),$$

are represented by L, A, B-triples with the following operators

$$A^+ = \partial^5 + V\partial^3 + V_z\partial^2 + W\partial,$$

$$B^+ = V_z\partial^2 + V_{zz}\partial + W_z,$$

$$A^- = \bar{A}^+, B^- = \bar{B}^+,$$

and

$$\bar\partial V = 5\partial U,$$

$$\bar\partial W = 5\partial^3 U + 3V\partial U + V_z U.$$

The second equations of the mNV hierarchy are more complicated :

$$U_{t_3^+} = \Phi_{mNV2}(U), \quad U_{t_3^-} = \overline{\Phi_{mNV2}(U)},$$

where

$$\Phi(U)_{mNV2} = U_{zzzzz} + 5VU_{zzz} + \frac{15}{2}V_z U_{zz} + 5(V^2 - \frac{3}{2}V_{zz} + W)U_z + 5(VV_z - V_{zzz} + \frac{1}{2}W_z))U$$

and

$$V_{\bar{z}} = (U^2)_z, W_{\bar{z}} = (U^2 V - U_z^2)_z.$$

Operators A and B are given by

$$A^+ = \partial^5 + 5\begin{pmatrix} 0 & -U_z \\ 0 & V \end{pmatrix}\partial^3 + 5\begin{pmatrix} 0 & UV - U_{zz} \\ 0 & \frac{3}{2}V_z \end{pmatrix}\partial^2 +$$

$$5\begin{pmatrix} 0 & \frac{1}{2}UV_z - U_z V - U_{zzz} \\ 0 & V^2 - \frac{3}{2}V_{zz} + W \end{pmatrix}\partial + 5\begin{pmatrix} 0 & UV^2 - 2UV_{zz} + U_{zz}V + \frac{1}{2}U_z V_z + UW \\ 0 & VV_z - V_{zzz} + W_z \end{pmatrix},$$

$$B^+ = 5\begin{pmatrix} 0 & U_z \\ -U_z & 0 \end{pmatrix}\partial^3 + 5\begin{pmatrix} 0 & U_{zz} - UV \\ -UV - 2U_{zz} & 0 \end{pmatrix}\partial^2 +$$

$$5\begin{pmatrix} 0 & U_z V + U_{zzz} - \frac{1}{2}UV_z \\ -\frac{3}{2}UV_z - 2U_{zzz} - 3U_z V & 0 \end{pmatrix}\partial -$$

$$5\begin{pmatrix} 0 & U(V^2 + W - 2V_{zz}) + U_{zz}V + \frac{1}{2}U_z V_z \\ U(V^2 + W - \frac{3}{2}V_{zz}) + U_{zz}V + 3U_z V_z + U_{zzzz} & 0 \end{pmatrix},$$

and

$$A^- = \bar{\partial}^5 + 5\begin{pmatrix} \bar{V} & 0 \\ U_{\bar{z}} & 0 \end{pmatrix}\bar{\partial}^3 + 5\begin{pmatrix} \frac{3}{2}\bar{V}_{\bar{z}} & 0 \\ -U\bar{V} + U_{\bar{z}\bar{z}} & 0 \end{pmatrix}\bar{\partial}^2 +$$

$$5\begin{pmatrix} \bar{V}^2 - \frac{3}{2}\bar{V}_{\bar{z}\bar{z}} + \bar{W} & 0 \\ -\frac{1}{2}U\bar{V}_{\bar{z}} + U_{\bar{z}}V + U_{\bar{z}\bar{z}\bar{z}} & 0 \end{pmatrix}\bar{\partial} + 5\begin{pmatrix} \bar{V}\bar{V}_{\bar{z}} - \bar{V}_{\bar{z}\bar{z}\bar{z}} + \bar{W}_{\bar{z}} & 0 \\ -U\bar{V}^2 + 2U\bar{V}_{\bar{z}\bar{z}} - U_{\bar{z}\bar{z}}\bar{V} - \frac{1}{2}U_{\bar{z}}\bar{V}_{\bar{z}} - U\bar{W} & 0 \end{pmatrix},$$

$$B^- = 5\begin{pmatrix} 0 & U_{\bar{z}} \\ -U_{\bar{z}} & 0 \end{pmatrix}\bar{\partial}^3 + 5\begin{pmatrix} 0 & U\bar{V} + 2U_{\bar{z}\bar{z}} \\ -U_{\bar{z}\bar{z}} + U\bar{V} & 0 \end{pmatrix}\partial^2 +$$

$$5\begin{pmatrix} 0 & \frac{3}{2}U\bar{V}_{\bar{z}} + 2U_{\bar{z}\bar{z}\bar{z}} + 3U_{\bar{z}}\bar{V} \\ -U_{\bar{z}}\bar{V} - U_{\bar{z}\bar{z}\bar{z}} + \frac{1}{2}U\bar{V}_{\bar{z}} & 0 \end{pmatrix}\partial +$$

$$5\begin{pmatrix} 0 & U(\bar{V}^2 + \bar{W} - \frac{3}{2}\bar{V}_{\bar{z}\bar{z}}) + U_{\bar{z}\bar{z}}\bar{V} + 3U_{\bar{z}}\bar{V}_{\bar{z}} + U_{\bar{z}\bar{z}\bar{z}\bar{z}} \\ U(\bar{V}^2 + \bar{W} - 2\bar{V}_{\bar{z}\bar{z}}) + U_{\bar{z}\bar{z}}\bar{V} + \frac{1}{2}U_{\bar{z}}\bar{V}_{\bar{z}} & 0 \end{pmatrix}$$

Equations

$$U_t = \Phi_{NV2}(U) + \overline{\Phi_{NV2}(U)}$$

and

$$U_t = \Phi_{mNV2}(U) + \overline{\Phi_{mNV2}(U)}$$

preserve reality of potentials analogously to the case of the first equations.

3. WEIERSTRASS REPRESENTATION

3.1. Construction of minimal surfaces.

The most general method of constructing minimal surfaces in the three-dimensional Euclidean space was introduced by Weierstrass, and we will start with it our explanation of representation of surfaces.

Let take a pair of functions (ψ_1, ψ_2) such that one of them, ψ_1, is antiholomorphic and another, ψ_2, is holomorphic. Let us suppose that these functions are

defined at the same simply connected domain S in a complex plane. We have a system of equations

$$\begin{cases} \psi_{1z} = 0, \\ \psi_{2\bar{z}} = 0. \end{cases} \tag{3.1}$$

Let us now define in terms of these functions a mapping

$$T : S \to \mathbf{R}^3 \tag{3.2}$$

by the following formulas

$$z \in S \to (X^1(z, \bar{z}), X^2(z, \bar{z}), X^3(z, \bar{z})) \in \mathbf{R}^3$$

where

$$X^1 + iX^2 = i \int_\gamma (\bar{\psi}_1^2 dz' - \bar{\psi}_2^2 d\bar{z}'),$$

$$X^1 - iX^2 = i \int_\gamma (\psi_2^2 dz' - \psi_1^2 d\bar{z}'), \tag{3.3}$$

$$X^3 = - \int_\gamma (\psi_2 \bar{\psi}_1 dz' + \psi_1 \bar{\psi}_2 d\bar{z}').$$

Everywhere we suppose that integrals are taken over any path γ which lies in the domain S and connects point z with some initial point z_0. It follows from (3.1) that integrands are closed forms and hence values of integrals do not depend on choice of path γ.

Weierstrass had shown that

surface $T(S)$ is minimal that means that it's mean curvature vanishes everywhere.

3.2. Generalized Weierstrass formulas.

It is naturally to ask when formulas (3.3) define a surface in the three-dimensional Euclidean space. As one can see integrands ought to be closed forms and this condition is sufficient. In the case of the Weierstrass representation that follows from (3.1).

It turns out that if functions ψ_1, ψ_2 satisfy more general system

$$\begin{cases} \psi_{1z} = U\psi_2, \\ \psi_{2\bar{z}} = -U\psi_1 \end{cases} \tag{3.4}$$

with real potential U then integrands in (3.3) occur to be closed forms. Hence in this case formulas (3.3) define a surface for every solution to system (3.4).

That was shown in [Kon] where formulas for induced metric and curvatures were also derived. Let us explain them here. Coordinates $(z, \bar{z})$ are conformal and in terms of them the metric tensor takes the form

$$D(z, \bar{z})^2 dz d\bar{z} \tag{3.5}$$

where

$$D(z,\bar{z}) = |\psi_1(z,\bar{z})|^2 + |\psi_2(z,\bar{z})|^2.$$

The Gaussian curvature is given by

$$K = -\frac{1}{D^2}\Delta \log D, \qquad (3.6)$$

and the mean curvature takes the form

$$H = \frac{2U}{D}. \qquad (3.7)$$

This representation is not new . For instance, it is given in survey [Bb], it was discussed by U. Abresch in middle 80's with it's relation to constructing constant mean curvature surfaces, in other terms it is given in a book of Eisenhart ([E]), and moreover it is equivalent to the well-known Kenmotsu representation ([Ken], see also [HO]).

We will show in 3.4 that it is almost equivalent to the definition of the second fundamental form (see Proposition 1). Notice that the convenience of this form of representation is that operator in linear problem (3.4) coincides with the operator L^{mNV} to which the modified Novikov-Veselov hierarchy is attached. That was the main source for definition of local deformation of surfaces which was given in [Kon] where this representation was rediscovered.

3.3. On representation of surfaces by Weierstrass formulas.

Let us consider the question how wide is the class of surfaces represented by formulas (3.3 - 3.4).

Let

$$F : \Sigma \to \mathbf{R}^3 \qquad (3.8)$$

be a regular mapping of the domain Σ, of the complex plane $\mathbf{C}$ with coordinates $(z, \bar{z})$, into the three-dimensional Euclidean space, and the induced metric is conformally Euclidean with respect to these coordinates, i.e. a metric tensor takes the form $D(z, \bar{z})^2 dz d\bar{z}$.

In this case the vector

$$G(z) = \left(\frac{\partial F^1}{\partial z}, \frac{\partial F^2}{\partial z}, \frac{\partial F^3}{\partial z}\right)$$

satisfies evident equation

$$\left(\frac{\partial F^1}{\partial z}\right)^2 + \left(\frac{\partial F^2}{\partial z}\right)^2 + \left(\frac{\partial F^3}{\partial z}\right)^2 = 0. \qquad (3.9)$$

That immediately follows from the following formula

$$G(z) = \frac{\partial F}{\partial z} = \frac{1}{2}\left(\frac{\partial F}{\partial x} - i\frac{\partial F}{\partial y}\right) \qquad (3.10)$$

and the condition that the metric is conformally Euclidean :

$$\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial x}\right) = \left(\frac{\partial F}{\partial y}, \frac{\partial F}{\partial y}\right), \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right) = 0.$$

The subvariety $Q \subset \mathbf{C}P^2$, which is defined in terms of homogeneous coordinates $(\varphi_1, \varphi_2, \varphi_3)$ by equation

$$\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 0,$$

is diffeomorphic to the Grassmann manifold $G_{3,2}$ formed by two-dimensional sub-spaces of $\mathbf{R}^3$. This diffeomorphism is given by the mapping

$$G_{3,2} \to Q$$

which corresponds to a plane, generated by a pair of orthogonal unit vectors (a_1, a_2, a_3) and (b_1, b_2, b_3), a point $(a_1 + ib_1, a_2 + ib_2, a_3 + ib_3) \in Q$.

Thus we can consider this mapping G as the Gauss map.

The Gauss map, defined in this manner, for surface (3.3) takes the form

$$G(z) = (i(\bar{\psi}_1^2 + \psi_2^2)/2, (\bar{\psi}_1^2 - \psi_2^2)/2, -\psi_2\bar{\psi}_1). \tag{3.11}$$

This formula gives us an idea how to prove the following Proposition.

Proposition 1. *Every regular conformally Euclidean immersion of surface into the three-dimensional Euclidean space is locally defined by generalized Weierstrass formulas (3.3 - 3.4).*

For the sake of brevity, we did not mention that formulas (3.3 - 3.4) represent locally every surface up to translation in $\mathbf{R}^3$. That is easy to see from (3.3).

Proof of Proposition 1.

We assume that $F_z^3 \neq 0$ otherwise change coordinates in $\mathbf{R}^3$ to get that.

Let us compare (3.10) and (3.11) and define functions

$$\begin{cases} \varphi_1 = \sqrt{F_{\bar{z}}^2 + iF_{\bar{z}}^1}, \\ \varphi_2 = \sqrt{-(F_z^2 + iF_z^1)}. \end{cases} \tag{3.12}$$

It follows from (3.9) that

$$F_z^3 = -\bar{\varphi}_1\varphi_2.$$

Let us now remind the definition of the second fundamental form h_{ij}. Let $D(z, \bar{z})^2 dz d\bar{z}$ be a metric tensor on surface (3.8). We take in the tangent plane (at point z) an orthonormal basis

$$e_1 = \frac{1}{D}\frac{\partial F}{\partial x}, \qquad e_2 = \frac{1}{D}\frac{\partial F}{\partial y}$$

and extend it to a basis in $\mathbf{R}^3$ by adding a unit normal vector

$$e_3 = e_1 \times e_2.$$

Components of the curvature tensor are defined by the well-known decomposition formulas (see, for instance, [Ken]):

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial D}{\partial x} e_1 - \frac{\partial D}{\partial y} e_2 + D^2 h_{11} e_3,$$

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial D}{\partial y} e_1 + \frac{\partial D}{\partial x} e_2 + D^2 h_{12} e_3,$$

$$\frac{\partial^2 F}{\partial y^2} = -\frac{\partial D}{\partial x} e_1 + \frac{\partial D}{\partial y} e_2 + D^2 h_{22} e_3.$$

Substitute these expressions for second derivatives of F into formulas for $\varphi_{1z}, \varphi_{2\bar{z}}$, derived from (3.12), and by direct computations obtain

$$\begin{cases} \varphi_{1z} = \frac{DH}{2} \varphi_2, \\ \varphi_{2\bar{z}} = -\frac{DH}{2} \varphi_1, \end{cases}$$

where H is a mean curvature.

Proposition 1 is proved.

The important corollary of proposition 1 is the following Proposition.

Proposition 2. *Every regular analytic surface is represented by formulas (3.3 - 3.4) globally.*

That follows from existence of local representation and unique analytic continuation.

3.4. Examples of surfaces represented by Weierstrass formulas.

Let us consider the simplest examples of surfaces represented by formulas (3.3 - 3.4).

1) **Surfaces of revolution.**

We assume, without loss of generality, that the axis OX^3 is the axis of revolution. In this case functions ψ_1 and ψ_2 are given by

$$\psi_1 = r_1(x) \exp \frac{iy}{2}, \qquad \psi_2 = r_2(x) \exp \frac{iy}{2},$$

and system (3.4) takes the form

$$\left(\frac{\partial}{\partial x} - \frac{1}{2} \begin{pmatrix} -1 & 4U \\ -4U & 1 \end{pmatrix} \right) \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = 0. \tag{3.13}$$

Here a potential U depends only on one variable x and it is easy to see that the matrix differential operator from linear problem (3.13) coincides with the operator L^{mKDV} of form (2.10). Hence, in terms of the generalized Weierstrass representation, the reduction of L^{mNV} to L^{mKDV} has natural geometrical meaning.

2) **Closed surfaces with genus ≥ 1.**

Let $F : \Sigma \to \mathbf{R}^3$ be an immersion, of surface with genus $g \geq 1$, given by formulas (3.3 - 3.4).

It is well-known that every closed oriented surface Σ with positive genus is uniformizable that means that there exists a mapping

$$p : M \to \Sigma$$

of simply connected surface M with constant curvature (the Euclidean plane for $g = 1$ and the Lobachevsky plane for $g > 1$) which is conformal covering.

In other words there exists a discrete subgroup Γ of a group of isometries of M such that a factor–space M/Γ is conformally equivalent to the surface Σ.

We consider cases $g = 1$ and $g > 1$ separately.

2.1) **Tori ($g = 1$).**

In this case a subgroup Γ is isomorphic to a free Abelian group with rank 2 (i.e., two-dimensional lattice) generated by a pair of independent shifts.

If $\gamma \in \Gamma$ then

$$\gamma^*(dz d\bar{z}) = dz d\bar{z}$$

and hence the following Proposition holds.

Proposition 3. *Let Σ be a two-dimensional torus immersed into $\mathbf{R}^3$ by formulas (3.3 - 3.4). Then there exists a lattice of periods Γ, with rank 2, such that potential $U(z)$, metric tensor $D(z)^2$, and mean curvature are invariant with respect to action of Γ. Functions ψ_1, ψ_2 at the same time are transformed as follows*

$$\psi_1(z + \gamma) = (\pm 1)\psi_1(z),$$

$$\psi_2(z + \gamma) = (\pm 1)\psi_2(z),$$

$$z \to z + \gamma, \quad \gamma \in \Gamma.$$

2.2) **Surfaces with genus $g > 1$.**

In this case a space M is isometric to the upperhalf plane $\mathcal{H} = \{(x + iy) \in \mathbf{C} | y > 0\}$ endowed with the metric $(dx^2 + dy^2)/y^2$. The group of isometries of $\mathcal{H}$ is the group $PSL(2, \mathbf{R})$ which acts by fractional linear transformations

$$z \to \frac{az + b}{cz + d},$$

$$a, b, c, d \in \mathbf{R}, \quad ad - bc = 1.$$

The action of elements of $PSL(2, \mathbf{R})$ on differentials takes the form

$$\gamma^*(dz) = \frac{dz}{(cz + d)^2}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We conclude that

$$D(\gamma(z)) = |cz + d|^2 D(z). \tag{3.14}$$

Since the mean curvature is invariant ($H(z) = H(\gamma(z))$), it follows from (3.7) that

$$U(\gamma(z)) = |cz + d|^2 U(z). \tag{3.15}$$

Now we are able to make the following conclusion.

Proposition 4. *Let a surface* Σ, *with genus* $g > 1$, *is immersed into* $\mathbf{R}^3$ *by formulas (3.3 - 3.4) and is conformally equivalent to a surface* $\mathcal{H}/\Gamma$ *where* Γ *is a discrete subgroup of* $PSL(2, \mathbf{R})$. *Then metric tensor* $D(z)^2$ *and potential* $U(z)$ *are transformed by elements of* Γ *by formulas (3.14) and (3.15) respectively, and* Γ *acts on functions* ψ_1 *and* ψ_2 *as follows*

$$\psi_1(\gamma(z)) = (c\bar{z} + d)\psi_1(z),$$

$$\psi_2(\gamma(z)) = (cz + d)\psi_2(z).$$

4. Deformation of surfaces by the modified Novikov-Veselov equation

4.1. Definition of deformation.

In paper [Kon] Konopelchenko by using of representation (3.3 - 3.4) defined a new class of deformations of surfaces. The mean observation of this paper is that the operator from the linear problem (3.4) coincides with the operator L^{mNV} to which the modified Novikov-Veselov hierarchy is attached. Hence the following deformation is naturally defined:

1) let $F : S \to \mathbf{R}^3$ be a surface immersed by formulas (3.3 - 3.4) ;

2) assume that the potential $U(z, \bar{z}, t)$ is being transformed in t via the modified Novikov-Veselov equation (2.11). At the same time eigenfunctions ψ_1, ψ_2 are being transformed via equation (2.17) and that generates deformation of surface F.

We call this deformation *modified Novikov-Veselov deformation* (mNV deformation).

Moreover it is stated in [Kon] that every equation from the mNV hierarchy generate deformation of this type. Minimal surfaces correspond to zero potentials and hence are stationary with respect to these flows.

An interesting observation was done in [KT] where the following Proposition was proved.

Proposition 5. *([KT]) 1) An integral of squared mean curvature over closed immersed (via (3.3 - 3.4)) surface* S , *i.e. the value of the Willmore functional at surface* S, *is equal to*

$$W(S) = 4 \int_{F(S)} U(z, \bar{z})^2 dz d\bar{z} \quad ; \tag{4.1}$$

2) If closed surface $F(S)$ *is deformed by the mNV flow into closed surfaces and lattices of periods of functions* U *and* V *are preserved then the value of the Willmore functional is also preserved.*

The proof of the first statement follows immediately from formulas (3.5) and (3.7). The second statement is derived from the following formula

$$UU_t = (UU_{zz} - \frac{U_z^2}{2} + \frac{3}{2}U^2 V)_z + (UU_{\bar{z}\bar{z}} - \frac{U_{\bar{z}}^2}{2} + \frac{3}{2}U^2 \bar{V})_{\bar{z}}, \tag{4.2}$$

which itself follows from equation (2.11).

In the spirit of this Proposition it is natural to study global properties of the mNV flow and it's relation to the theory of Willmore surfaces. In [KT] such investigation was started for tori of revolution.

We will not dwell here on results of [KT] on tori of revolution and explain here more general facts.

4.2. Global deformations of closed surfaces.

In this subchapter we consider the question when the mNV flow transforms closed surfaces into closed ones preserving their conformal structure.

First we thought that non-automorphic form of L^{mNV} and A operators (see Proposition 4) implies non-existence of mNV-deformations , of surfaces with genus $g \geq 2$, that preserve their closedness and conformal structure. But now we are persuaded by F. Pedit and U. Pinkall that that strongly depends on the correct understanding of constraint (2.12) and definition of V. Probably at least deformation of periodic Gauss map can be obtained in this manner. This problem is still in question and thus we will discuss the case of surfaces with higher genus elsewhere.

Thus we restrict ourselves by deformations of tori. First we prove the following Proposition.

Proposition 6. *There exists procedure which uniquely corresponds to a smooth double-periodic potential $U(z)$ a function $V(z)$ which satisfies (2.12).*

Proof of Proposition 6. Let $U(z)$ be a double-periodic function with a lattice of periods Γ and Γ^* be a lattice dual to Γ.

Any smooth function on the torus $\mathbf{C}/\Gamma = \mathbf{R}^2/\Gamma$ is decomposed into Fourier series with respect to basis formed by eigenfunctions of the operator $\bar{\partial}$. Notice that these functions also form a basis of eigenfunctions of the operator ∂. These eigenfunctions are of the form $f(z|g^*) = \exp 2\pi i g^*(z)$ where by $\gamma^*(z)$ we mean a scalar product of γ^* and vector $(Re\ z, Im\ z)$. For the sake of brevity we use only z as argument but it is easy to notice that these functions are not holomorphic.

It is evident that for double-periodic function $w(z)$ there exists double-periodic function $v(z)$ such that $v_{\bar{z}} = w$ if and only if the Fourier series for $w(z)$

$$w(z) = \sum_{g^* \in \Gamma^*} w_{\gamma^*} f(z|\gamma^*)$$

does not contain terms corresponding to the kernel of operator $\bar{\partial}$ (i.e., function $f(z|0) = 1$). In this case we can invert operator $\bar{\partial}$ evidently by using of the Fourier decomposition.

If the function $w(z)$ is a derivative of the double-periodic function itself then it's Fourier decomposition does not contain such term. Let put $w(z) = (U^2)_z$ and take a function $V(z) = \bar{\partial}^{-1} w(z)$ uniquely determined by additional condition

$$\int_{\mathbf{C}/\Gamma} V(z) dz d\bar{z} = 0.$$

This condition holds if and only if the Fourier series for $V(z)$ does not contain terms which lie in the kernel of $\bar{\partial}$.

Proposition 6 is proved.

Notice that if we will add to $V(z,t)$ a function which depends only on t when we will not change geometric deformation of surface but only include linear translation of conformal coordinates $(z, \bar{z})$.

Let us consider two integrals

$$\frac{\partial(X^1(z,t) + iX^2(z,t))}{\partial t} = 2i \int \Omega_0$$

and

$$\frac{\partial X^3}{\partial t} = - \int \Omega_1$$

where

$$\Omega_0 = \frac{1}{2}((\psi_2^2)_t dz - (\psi_1^2)_t d\bar{z}),$$

$$\Omega_1 = (\psi_{2t}\bar{\psi}_1 + \psi_2\bar{\psi}_{1t})dz + (\psi_{1t}\bar{\psi}_2 + \psi_1\bar{\psi}_{2t})d\bar{z}.$$

Explicit formulas for differentials Ω_0 and Ω_1 follows from (2.17). We omit them also as rather large computations which need only formulas (2.12) and (3.3) and give the following result.

Proposition 7.

$$1) \quad \Omega_0 = d(f_1 + g_1 + f_2 + g_2)$$

where

$$f_1 = \frac{3}{2}V\psi_2^2, \qquad g_1 = \psi_2\partial^2\psi_2 - \frac{(\partial\psi_2)^2}{2},$$

$$f_2 = \frac{3}{2}\bar{V}\psi_1^2, \qquad g_1 = \psi_1\bar{\partial}^2\psi_1 - \frac{(\bar{\partial}\psi_1)^2}{2};$$

$$2) \quad \Omega_1 = d(h_1 + h_2)$$

where

$$h_1 = \bar{\psi}_1\partial^2\psi_2 + \psi_2\partial^2\bar{\psi}_1 - \partial\psi_2\partial\bar{\psi}_1 + 3V\bar{\psi}_1\psi_2,$$

$$h_2 = \psi_1\bar{\partial}^2\bar{\psi}_2 + \bar{\psi}_2\bar{\partial}^2\psi_1 - \bar{\partial}\bar{\psi}_2\bar{\partial}\psi_1 + 3\bar{V}\psi_1\bar{\psi}_2.$$

Moreover two modified Novikov-Veselov deformations generated by L, A, B-triples (2.14) and (2.15) satisfy to the following formal equations

$$\frac{\partial(X^1(z,t^+) + iX^2(z,t^+))}{\partial t^+} = 2i \int d(f_1 + g_1),$$

$$\frac{\partial(X^1(z,t^-) + iX^2(z,t^-))}{\partial t^-} = 2i \int d(f_2 + g_2).$$

which can be useful for proving of analogues of Proposition 7 for deformations generated by higher equations of the mNV hierarchy.

Now we can formulate the main Theorem.

Theorem 1. *Let Σ be a two-dimensional torus represented by formulas (3.3 - 3.4) with double-periodic potential $U(z)$, and let $U(z,t)$ be a solution to equation (2.11) with initial data $U(z,0) = U(z)$ and double-periodic potential $V(z,t)$. Then the mNV flow deforms torus Σ into tori Σ_t which are represented by (3.3 - 3.4) with potentials $U(z,t)$, conformally equivalent to Σ and have the same value of the Willmore functional.*

Proof of Theorem 1.

By Proposition 7, forms Ω_0 and Ω_1 are exact on torus $\mathbf{C}/\Gamma$ being differentials of double-periodic functions. Therefore, a lattice of periods, which determines conformal class, is preserved by the mNV flow.

Now it follows from Proposition 5 that a value of the Willmore functional is also preserved.

Theorem 1 is proved.

In analytic case the stronger theorem holds.

Theorem 2. *The modified Novikov-Veselov equation induces via formulas (2.17) and (3.3 - 3.4) deformation of immersed analytic tori. Moreover this deformation preserves their conformal structures and values of the Willmore functional.*

Proof of Theorem 2.

It follows from Proposition 2 that every analytic torus is represented by formulas (3.3 - 3.4). Since tori are analytic and by Proposition 6 and the Cauchy-Kowalewski theorem a solution, of the modified Novikov-Veselov equation, which satisfies conditions of Theorem 1 exists at least for small t.

Now Theorem 2 follows from Theorem 1.

Theorem 2 is proved.

4.3. Clifford torus as stationary point of the mNV flow.

It follows from the definition of the potential $U(z)$ (see (3.7)) that geometrically stationary points of the mNV flow, i.e., surfaces which are transformed into images of itself with respect to translations in $\mathbf{R}^3$, correspond to stationary solutions of the mNV equation (2.11).

It is also naturally to expect that the simplest stationary solutions will be one-dimensional, i.e. stationary solutions of the modified Korteweg–de Vries equation.

We will show that the simplest stationary solution is realized by a prominent surface, *Clifford torus*.

Let S^3 be a unit sphere in the four-dimensional Euclidean space $\mathbf{R}^3$ with coordinates (x_1, x_2, x_3, x_4). The Clifford torus (in $\mathbf{R}^4$) is the image of the following embedded torus

$$\mathbf{R}^2 \to S^4 : (x,y) \to (\frac{\cos y}{\sqrt{2}}, \frac{\sin y}{\sqrt{2}}, \frac{\cos x}{\sqrt{2}}, \frac{\sin x}{\sqrt{2}}). \qquad (4.3)$$

Let us consider the stereographic projection of S^4 onto the plane $x^4 = -1$ with the pole $(0,0,0,1)$:

$$(x_1, x_2, x_3, x_4) \to (\frac{-2x_1}{x_4 - 1}, \frac{-2x_2}{x_4 - 1}, \frac{-2x_3}{x_4 - 1}, -1). \qquad (4.4)$$

We call the image of the Clifford torus with respect to this projection Clifford again.

The variables (x, y) occur to be conformal and the metric tensor takes the form

$$\frac{4}{(\sqrt{2} - \sin x)^2}(dx^2 + dy^2). \tag{4.5}$$

Gaussian and mean curvatures are given by

$$K = \frac{\sqrt{2}\sin x - 1}{4}, \quad H = \frac{\sin x}{2\sqrt{2}}. \tag{4.6}$$

Let us determine potential $U(x)$ by formula (3.7) :

$$U(x) = \frac{\sin x}{2\sqrt{2}(\sqrt{2} - \sin x)}. \tag{4.7}$$

It follows from direct computations that this potential induces the Clifford torus by formulas (3.3 - 3.4). Let us also notice that potential (4.7) satisfies the following equation

$$U_x^2 = -4U^4 + 2U^2 + \frac{U}{\sqrt{2}} + \frac{1}{16}. \tag{4.8}$$

If a solution of the mNV equation depends only on variable $x - const \cdot t$ then it satisfies to equation

$$(U_{xxx} + 24U^2 U_x - const \cdot U_x)_x = 0. \tag{4.9}$$

If follows from (4.8) that the Clifford torus (4.7) satisfies (4.9). Hence we conclude that the Clifford torus is a geometrically stationary point of the mNV flow.

5. WILLMORE FUNCTIONAL

We already mentioned above (see Proposition 5 and Theorems 1 and 2) that the mNV flow preserves values of the Willmore functional and briefly gave definition of this functional. Last years this functional attracted a lot of attention of geometers (see history of it's investigation and explanation of a lot of facts about it in [Wm], also see [ST,W,LY,Br,LS,Kus,FPPS,HJP,Sm,BBb,B]).

In this chapter we will try to get a brief survey of the modern history of the Willmore conjecture and consider it's relation to the mNV flow.

Let $F : S \to \mathbf{R}^3$ be an immersed surface. A value of the Willmore functional at this surface if defined by the following formula :

$$W(S) = \int_S H^2 d\mu. \tag{5.1}$$

Here $d\mu$ the Liouville measure with respect to the induced metric on S.

This functional is conformally invariant, i.e., any conformal transformation of the three-dimensional Euclidean space transforms any immersed surface into another one with the same value of the Willmore functional.

We call surface *Willmore* if this surface is a critical point of the Willmore functional. The Euler-Lagrange equation for this functional has the form

$$\Delta H + 2H(H^2 - K) = 0, \tag{5.2}$$

where Δ is the Laplace-Beltrami operator on surface ([Wm]).

The following Proposition can be obtained by using of direct computations.

Proposition 8. *If surface is represented by formulas (3.3 - 3.4) then it is Willmore if and only if the following equality holds*

$$\Delta U \cdot D - 2(U_x D_x + U_y D_y) + U \cdot \Delta D + 8U^3 D = 0 \tag{5.3}$$

where $z = x + iy$ and $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

The simplest examples of Willmore surfaces are stereographic projections of minimal surfaces M in S^3. Moreover an area of a minimal surface M in S^3 is equal to a value of the Willmore functional at it's projection. All that was known to Thomsen and Blaschke in 20-s. Conformal properties of this functional and it's relation to minimal surfaces gave Blaschke a reason to call such surfaces *conformally minimal* ([Bl]).

But these examples do not cover the class of Willmore surfaces (see, for instance, [P] where the first examples of compact embedded Willmore surfaces that are not stereographic projections of minimal surfaces in S^3 were obtained).

All Willmore spheres were classified by Bryant ([Br])

The main attention attracts the conjecture posed by Willmore in middle 60-s.

Willmore Conjecture. *For immersed tori the Willmore functional satisfies the following inequality*

$$W \geq 2\pi^2, \tag{5.4}$$

which is attained only for Clifford torus and it's images under conformal transformations of $\mathbf{R}^3$.

It's analogues for all genuses were posed in [Kus] but this conjecture is still open.

Simon proved that minimum is attained on an analytic minimal torus ([Sm]).

The following list contains all known classes of tori for which Willmore conjecture was proved.

1) In early 70-s Willmore and independently Shiohama and Takagi ([ST]) proved this conjecture for tube tori with constant radii. Here we call torus tube if is formed carrying a small circle round a closed space curve such that the center moves along the curve and the plane of the circle is the normal plane to the curve at each point.

2) Hertrich-Jeromin and Pinkall ([HJP]) generalized result of Willmore-Shiohama-Takagi for tube tori with arbitrary radii, i.e. radius of circle can vary along the curve.

3) Langer and Singer ([LS]) proved the Willmore conjecture for tori of revolution.

4) Li and Yau in paper [LY] that brings together the spectral theory of the Laplace-Beltrami operator with the theory of conformal invariants proved this conjecture for tori whose conformal structures are defined by lattices generated by vectors $(1,0)$ and (a,b) where

$$0 \leq a \leq \frac{1}{2}, \quad \sqrt{1 - a^2} \leq b \leq 1.$$

In terms of theta-functions all Willmore tori are described in [BBb,B] (see also [FPPS]). Regretfully theta-functional formulas are very complicated and not rather efficient for applications.

We propose the following conjecture.

Conjecture. *Non-stationary, with respect to the mNV flow, torus can not be a local minimum of the Willmore functional.*

As it seems to us this conjecture looks truly because it is strange to expect that minimum, of this variational problem, taken up to conformal transformations of $\mathbf{R}^3$ is degenerated. Probably methods developed in [W,Pm] will be helpful for proving it.

If this conjecture is true then the Willmore conjecture is reduced to investigation of stationary points of the mNV flow. It is known from the soliton theory that stationary solutions ought to be simpler than general ones. For instance, stationary solutions of equations from the KDV hierarchy are described by very simple hyperelliptic functions. Of course the mNV equation is $2+1$-equation and we can not expect for it so simple description.

We also would like to pose the following question.

Question. *Higher equations of the mNV hierarchy also have first integrals. What is a geometric meaning of critical points of these functionals ?*

Similarity of formulas for mNV and mNV-2 equations shows that it needs to expect that these flows will deform tori into tori. Thus these deformations ought to have geometric meaning. Most probably these flows preserve conformal structures and these flows have origin in conformal geometry.

References

[B]   Babich M.V., *Willmore surfaces, 4-particles Toda lattice and double coverings of hyperelliptic surfaces*, Preprint INS 249. Clarkson University (May 1994).

[BBb]   Babich M., Bobenko A., *Willmore tori with umbilic points and minimal surfaces in hyperbolic space*, Duke Math. Journal **72** (1993), 151–185.

[Bb]   Bobenko A.I., *Surfaces in terms of 2 by 2 matrices. Old and new integrable cases*, Harmonic Maps and Integrable Systems", Eds. Fordy A., Wood J., Vieweg, 1994, pp. 83–127.

[Bg1]   Bogdanov L.V., *Veselov–Novikov equation as a natural two-dimensional generalization of the Korteweg–de Vries equation*, Theor. Math. Phys. **70** (1987), 309–314.

[Bg2]   Bogdanov L.V., *On the two-dimensional Zakharov–Shabat problem*, Theor. Math. Phys. **72** (1987), 790–793.

[Bl]   Blaschke W., *Vorlesungen über Differentialgeometrie III*, Springer, Berlin, 1929.

[Br]   Bryant R., *A duality theorem for Willmore surfaces*, J. Diff. Geom. **20** (1984), 23–53.

[DKN]   Dubrovin B.A., Krichever I.M., Novikov S.P., *The Schrödinger equation in a periodic field and Riemann surfaces*, Soviet Math. Dokl. **17** (1976), 947–951.

[E]   Eisenhart L.P., *A treatise on the differential geometry of curves and surfaces*, Boston, Allyn and Bacon, 1909.

[FPPS]   Ferus D., Pedit F., Pinkall U., Sterling I., *Minimal tori in S^4*, J. Reine Angew. Math. **429** (1992), 1–47.

[FS]   Focas A.S., Santini P.M., *The recursion operator of the Kadomtsev–Petviashvili equation and the squared eigenfunctions of the Schrodinger operator*, Studies in Appl. Math. **75** (1986), 179–185.

[HJP]   Hertrich–Jeromin U., Pinkall U., *Ein Beweis des Willmoreschen Vermutung fur Kanaltori*, J. Reine Angew. Math. **430** (1992), 21–34.

[HO]   Hoffman D.A., Osserman R., *The Gauss map of surfaces in $\mathbf{R}^3$ and $\mathbf{R}^4$*, Proceedings of the London Math. Society **50** (1985), 27–56.

[Ken]   Kenmotsu K., *Weierstrass formula for surfaces of prescribed mean curvature*, Math. Ann. **245** (1979), 89–99.

[Kon]   Konopelchenko B.G., *Induced surfaces and their integrable dynamics*, Preprint INP 93–114. Novosibirsk (to appear in Studies in Applied Mathematics) (1993).

[KT]   Konopelchenko B.G., Taimanov I.A., *Generalized Weierstrass formulae, soliton equations and Willmore surfaces. I. Tori of revolution and the mKDV equation*, Preprint Ruhr-Universitat-Bochum. Fakultat fur Mathematik. Nr. 187. (to appear in Studies in Applied Mathematics) (August 1995).

[Kus]   Kusner R., *Comparison surfaces for the Willmore problem*, Pacific J. Math. **138** (1989), 317–345.

[LS]   Langer J., Singer D., *Curves in the hyperbolic plane and mean curvature of tori in 3–space*, The Bulletin of the London Math. Soc. **16** (1984), 531–534.

[LY]   Li P., Yau S.T., *A conformal invariant and applications to the Willmore conjecture and the first eigenvalue for compact surfaces*, Invent. Math. **69** (1982), 269–291.

[M]   Manakov S.V., *Method of inverse scattering and two-dimensional evolution equations*, Uspekhi matematicheskikh nauk **31 (5)** (1976), 245–246. (Russian)

[N]   Novikov S.P., Manakov S.V., Pitaevskii L.P., Zakharov V.E., *Theory of solitons*, Consultants Bureau, New York, 1984.

[P]   Pinkall U., *Hopf tori in S^3*, Invent. Math. **81** (1985), 379–386.

[Pm]   Palmer B., *Second variational formulas for Willmore surfaces*, Technische Universitat Berlin. SFB 288. Preprint no. 23 (1992).

[ST]   Shiohama K., Takagi A., *A characterization of a standard torus in E^3*, J. Diff. Geom. **4** (1970), 477–485.

[Sh]   Shiota T., *Prym Varieties and soliton equations*, Infinite dimensional Lie algebras and groups, Advanced Studies in Math. Physics, vol. 7. World Scientific, 1989.

[Sm]   Simon L., *Existence of surfaces minimizing the Willmore problem*, Comm. in Analysis and Geometry **1** (1993), 281–326.

[T]   Taimanov I.A., *On an analogue of Novikov's conjecture in a problem of Riemann-Schottky type for Prym varieties*, Soviet Math. Dokl. **35** (1987), 420–424.

[VN1]   Veselov A.P., Novikov S.P., *Finite-zone, two-dimensional potential Schrödinger operators. Explicit formulas and evolution equations*, Soviet Math. Dokl. **30** (1984), 588–591.

[VN2]   Veselov A.P., Novikov S.P., *Finite-zone, two-dimensional Schrödinger operators. Potential operators.*, Soviet Math. Dokl. **30** (1984), 705–708.

[W]   Weiner J., *On a problem of Chen, Willmore, et al.*, Indiana Univ. Math. J. **27** (1978), 19–35.

[Wm]   Willmore T.J., *Riemannian geometry*, Clarendon Press, Oxford, 1993.

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